

G'_p -SPACES FOR MAPS AND HOMOLOGY DECOMPOSITIONS

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ABSTRACT. For a map $p : X \rightarrow A$, we define and study a concept of G'_p -space for a map, which is a generalized one of a G' -space. Any G' -space is a G'_p -space, but the converse does not hold. In fact, CP^2 is a G'_δ -space, but not a G' -space. It is shown that X is a G'_p -space if and only if $G^n(X, p, A) = H^n(X)$ for all n . We also obtain some results about G'_p -spaces and homology decompositions for spaces. As a corollary, we can obtain a dual result of Haslam's result about G -spaces and Postnikov systems.

1. Introduction

The Gottlieb groups $G_n(X)$ of a space X have been defined by Gottlieb in [3, 4]. A space X is called a G -space if $G_n(X) = \pi_n(X)$ for all n . It is well known [4] that any H -space is a G -space, but the converse does not hold. On the other hand, Haslam in [6] introduced the dual Gottlieb groups $G^n(X)$ of a space X and the concepts of G' -spaces. A space X is called G' -space if $G^n(X) = H^n(X)$ for all n . It is known [6] that any co- H -space is a G' -space, but the converse does not hold. Moreover, Haslam studied relationships between Postnikov systems and G -spaces. In [6], He showed that if X is a G -space, then each X_n is G -space and all the k invariants k_X^{n+2} are G -primitive, and if X_{n-1} is a G -space and the k -invariants k_X^{n+1} is G -primitive, then X_n is a G -space.

In 1959, Eckmann and Hilton [2] introduced a dual concept of Postnikov system as follows; A *homology decomposition of X* consists of a sequence of spaces and maps $\{X_n, q_n, i_n\}$ satisfying (1) $q_n : X_n \rightarrow X$ induces an isomorphism $(q_n)_* : H_i(X_n) \rightarrow H_i(X)$ for $i \leq n$, (2) $i_n : X_n \rightarrow$

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X_{n+1} is a cofibration with cofiber $M(H_{n+1}(X), n)$ (a Moore space of type $(H_{n+1}(X), n)$), and (3) $q_n \sim q_{n+1} \circ i_n$. It is known by [7] that if X is a 1-connected space having the homotopy type of CW complex, then there is a homology decomposition $\{X_n, q_n, i_n\}$ of X such that $i_n : X_n \rightarrow X_{n+1}$ is the principal cofibration induced from $\iota : M(H_{n+1}(X), n) \rightarrow cM(H_{n+1}(X), n)$ by a map $\kappa'_n : M(H_{n+1}(X), n) \rightarrow X_n$ which is called the dual Postnikov invariants. A space X is called a *rational space* [14] if X is a 1-connected space having homotopy type of a CW-complex such that for each $n > 0$, $H_n(X, \mathbb{Z})$ is a finite dimensional vector space over \mathbb{Q} . It is well known [14] that if X and A are rational spaces and $p : X \rightarrow A$ is a based map, then there exist homology decompositions $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ for X and A respectively and induced maps $\{p_n : X_n \rightarrow A_n\}$ satisfying :

(1) for each n , the following diagram is homotopy commutative

$$\begin{array}{ccc} M(H_{n+1}(X), n) & \xrightarrow{\tilde{p}_*} & M(H_{n+1}(A), n) \\ \kappa'_n(X) \downarrow & & \kappa'_n(A) \downarrow \\ X_n & \xrightarrow{p_n} & A_n, \end{array}$$

that is, $(\kappa'_n(A), \kappa'_n(X)) : \tilde{p}_\# \rightarrow p_n$ is a map,

(2) $p_{n+1} : X_{n+1} \rightarrow A_{n+1}$ given by $p_{n+1} = \bar{p}_n$ satisfying commute diagram

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ i_n (= \iota_{\kappa'_n(X)}) \downarrow & & i'_n (= \iota_{\kappa'_n(A)}) \downarrow \\ X_{n+1} & \xrightarrow{p_{n+1}} & A_{n+1}, \end{array}$$

(3) for each n , the following diagram is homotopy commutative

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ q_n \downarrow & & q'_n \downarrow \\ X & \xrightarrow{p} & A. \end{array}$$

For a map $p : X \rightarrow A$, the dual Gottlieb sets $G^n(X, p, A)$ of a map $p : X \rightarrow A$, which are generalized of dual Gottlieb groups $G^n(X)$, are defined in [20]. In general, $G^n(X) \subset G^n(X, p, A) \subset H^n(X)$ for any map $p : X \rightarrow A$.

In this paper, for a map $p : X \rightarrow A$, we define and study a concept of G'_p -space for a map, which is a generalized one of a G' -space. Any G' -space is a G'_p -space, but the converse does not hold. In fact, CP^2

is a G'_δ -space, but not a G' -space. It is shown that X is a G'_p -space if and only if $G^n(X, p, A) = H^n(X)$ for all n . It is clear that any co- H^p -space is a co- T^p -space and any co- T^p -space is a G'_p -space. Moreover, we show that X is a G' -space if and only if for any space A and any map $p : X \rightarrow A$, X is a G'_p -space for a map $p : X \rightarrow A$. We can obtain the following results about G'_p -spaces and homology decompositions which are dual generalizations of the above Haslam's results about G -spaces and Postnikov systems. Let X and A be rational spaces and $p : X \rightarrow A$ a map, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ homology decompositions for X and A respectively. (1) If X is a G'_p -space for a map $p : X \rightarrow A$, then each X_n is G'_{p_n} -space and the all pairs of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \rightarrow p_n$ are G'_{p_n} -primitive. (2) If X_n is a G'_{p_n} -space and the pair of k' -invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \rightarrow p_n$ is G'_{p_n} -primitive, then X_n is a G'_{p_n} -space. As a corollary, we can obtain a result for G' -spaces as follows. Let X be rational space and $\{X_n, q_n, i_n\}$ homology decomposition for X . (1) If X is a G' -space, then each X_n is G' -space and all the k' invariants $k'_n(X)$ are G' -primitive. (2) If X_{n-1} is a G' -space and the k' -invariants $k'_n(X)$ is G' -primitive, then X_n is a G' -space.

Throughout this paper, space means a space of the homotopy type of connected locally finite CW complex. We assume also that spaces have non-degenerate base points. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by $*$. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \rightarrow Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta : X \rightarrow X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$, the folding map $\nabla : X \vee X \rightarrow X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. ΣX denote the reduced suspension of X and ΩX denote the based loop space of X . The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ . The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$ and $\tau(1_{\Sigma X})$ respectively.

2. G'_p -spaces for maps

Let $p : X \rightarrow A$ be a map. A based map $f : X \rightarrow B$ is called p -cocyclic [13] if there is a map $\theta : X \rightarrow A \vee B$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc}
 X & \xrightarrow{\theta} & A \vee B \\
 \Delta \downarrow & & j \downarrow \\
 X \times X & \xrightarrow{(p \times f)} & A \times B,
 \end{array}$$

where $j : A \vee B \rightarrow A \times B$ is the inclusion and $\Delta : X \rightarrow X \times X$ is the diagonal map. We call such a map θ a *coassociated map* of a p -cocyclic map f .

In the case $p = 1_X : X \rightarrow X$, $f : X \rightarrow B$ is called *cocyclic* [16]. Clearly any cocyclic map is a p -cocyclic map and also $f : X \rightarrow B$ is p -cocyclic iff $p : X \rightarrow A$ is f -cocyclic. The *dual Gottlieb set* $DG(X, p, A; B)$ for a map $p : X \rightarrow A$ [20] is the set of all homotopy classes of p -cocyclic maps from X to B . In the case $p = 1_X : X \rightarrow X$, we called such a set $DG(X, 1, X; B)$ the *dual Gottlieb set* [16] denoted $DG(X; B)$, that is, the dual Gottlieb set is exactly same with the dual Gottlieb set for the identity map. We denote $DG(X, p, A; K(\pi, n))$ by $G^n(X, p, A; \pi)$ and $DG(X, p, A; K(\mathbb{Z}, n))$ by $G^n(X, p, A)$, $DG(X; K(\mathbb{Z}, n))$ by $G^n(X)$. Haslam [6] introduced and studied the *coevaluation subgroups* $G^n(X; \pi)$ of $H^n(X; \pi)$. $G^n(X; \pi)$ is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\pi, n)$. A space X is called [6] a G' -space if $G^n(X) = H^n(X)$ for all n . The next proposition is an immediate consequence from the definition.

PROPOSITION 2.1. [22]

- (1) For any maps $g : X \rightarrow A$, $h : A \rightarrow B$ and any space C , $DG(X, g, A; C) \subset DG(X, hg, B; C)$.
- (2) $DG(X, B) = DG(X, 1_X, X; B) \subset DG(X, g, A; B) \subset DG(X, *, A; B) = [X, B]$ for any spaces X, A and B .
- (3) $DG(X, B) = \cap \{DG(X, g, A; B) | g : X \rightarrow A \text{ is a map and } A \text{ is a space}\}$.
- (4) If $h : A \rightarrow B$ is a homotopy equivalence, then $DG(X, g, A; C) = DG(X, hg, B; c)$.
- (5) For any map $k : Y \rightarrow X$, $k^*(DG(X, g, A; B)) \subset DG(Y, gk, A; B)$.
- (6) For any map $k : Y \rightarrow X$, $k^*(DG(X; B)) \subset DG(Y, k, X; B)$.
- (7) For any map $s : B \rightarrow C$, $s_*(DG(X, g, A; B)) \subset DG(X, g, A; C)$.

In general, $DG(X; B) \subset DG(X, p, A; B) \subset [X, B]$ for any map $p : X \rightarrow B$ and any space B . It is known [20] that for any n , $G^n(S^n \times S^n; \mathbb{Z}) \neq G^n(S^n \times S^n, p_1, S^n; \mathbb{Z}) \neq H^n(S^n \times S^n; \mathbb{Z})$.

A based map $g : X \rightarrow A$ is called *weakly cocyclic* [18] if $g^*(H^n(X)) \subset G^n(X)$ for all n . Any cocyclic map is an weakly cocyclic map, but the

converse does not holds. It is known [6] that RP^2 is a G' -space, but not co- H -space. Thus we know that the identity map 1_{RP^2} is an weakly cocyclic map, but not cocyclic map.

PROPOSITION 2.2. [18] X is a G' -space if and only if $e' : X \rightarrow \Omega\Sigma X$ is weakly cocyclic.

DEFINITION 2.3. Let $p : X \rightarrow A$ be a based map. A based map $g : X \rightarrow B$ is called an weakly p -cocyclic if $g^*(H^n(B)) \subset G^n(X, p, A)$ for all n .

The next proposition is an immediate consequence from the definition.

PROPOSITION 2.4. (1) If $g : X \rightarrow B$ is an weakly cocyclic map and $\theta : B \rightarrow C$ is an arbitrary map, then $\theta g : X \rightarrow C$ is weakly cocyclic.

(2) For a map $p : X \rightarrow A$, any weakly cocyclic map $g : X \rightarrow B$ is weakly p -cocyclic.

(3) For a map $p : X \rightarrow A$, if $g : X \rightarrow B$ is an weakly p -cocyclic map and $\theta : B \rightarrow C$ is an arbitrary map, then $\theta g : X \rightarrow C$ is weakly p -cocyclic.

The following proposition says that co- H -spaces are completely characterized by the dual Gottlieb sets.

PROPOSITION 2.5. [11] X is a co- H -space if and only if $DG(X, B) = [X, B]$ for any space B .

A space X is called [22] a co- H^p -space for a map $p : X \rightarrow A$ if there is a map $\theta : X \rightarrow X \vee A$ such that $j\theta \sim (1 \times p)\Delta$, where $j : X \vee A \rightarrow X \times A$ is the inclusion and $\Delta : X \rightarrow X \times X$ is the diagonal map, that is, $1_X : X \rightarrow X$ is p -cocyclic.

PROPOSITION 2.6. [22] X is a co- H^p -space for a map $p : X \rightarrow A$ if and only if $DG(X, p, A; B) = [X, B]$ for any space B .

A space X is called a co- T -space [18] if $e' : X \rightarrow \Omega\Sigma X$ is cocyclic. The following proposition says that co- T -spaces are completely characterized by the dual Gottlieb sets.

PROPOSITION 2.7. [18] X is a co- T -space if and only if $DG(X, \Omega B) = [X, \Omega B]$ for any space B .

A space X is called [23] a co- T^p -space for a map $p : X \rightarrow A$ if there is a map $\theta : X \rightarrow \Omega\Sigma X \vee A$ such that $j\theta \sim (e' \times p)\Delta$, where

$j : \Omega\Sigma X \vee A \rightarrow \Omega\Sigma X \times A$ is the inclusion and $\Delta : X \rightarrow X \times X$ is the diagonal map, that is, $e' : X \rightarrow \Omega\Sigma X$ is p -cocyclic.

PROPOSITION 2.8. [23] X is a $co-T^p$ -space for a map $p : X \rightarrow A$ if and only if $DG(X, p, A; \Omega B) = [X, \Omega B]$ for any space B .

It is clear, from Proposition 2.1(2) and the above propositions, that any $co-T$ -space is a $co-T^p$ -space for any map $p : X \rightarrow A$.

DEFINITION 2.9. A space X is called a G'_p -space for a map $p : X \rightarrow A$ if $e' : X \rightarrow \Omega\Sigma X$ is weakly p -cocyclic.

The following theorem says that a G'_p -space can be characterized by the dual Gottlieb sets for a map $p : X \rightarrow A$.

THEOREM 2.10. X is a G'_p -space for a map $p : X \rightarrow A$ if and only if $G^n(X, p, A) = H^n(X)$ for all n .

Proof. Suppose that X is a G'_p -space for a map $p : X \rightarrow A$. Let $g : X \rightarrow K(\mathbb{Z}, n) = K(\mathbb{Z}, n+1)$ be any map. Since $g = \Omega\tau^{-1}(g)e' : X \rightarrow \Omega K(\mathbb{Z}, n+1)$ and $e' : X \rightarrow \Omega\Sigma X$ is weakly p -cocyclic, $g : X \rightarrow K(\mathbb{Z}, n)$ is weakly p -cocyclic. On the other hand, suppose that $G^n(X, p, A) = H^n(X)$ for all n . Since $1_X : X \rightarrow X$ is weakly p -cocyclic, we know that the map $e' = e'1_X$ is weakly p -cocyclic and X is a G'_p -space for a map $p : X \rightarrow A$. □

Since $G^n(X) = DG(X; K(\mathbb{Z}, n)) \subset DG(X, p, A; K(\mathbb{Z}, n)) = G^n(X, p, A) \subset [X, K(\mathbb{Z}, n)] = H^n(X)$, any G' -space is a G'_p -space for any map $p : X \rightarrow A$.

Moreover, we can easily obtain, from the fact $K(\mathbb{Z}, n) \simeq \Omega K(\mathbb{Z}, n+1)$ and Proposition 2.6 and Proposition 2.8, the following corollary.

COROLLARY 2.11. Any $co-H^p$ -space is a $co-T^p$ -space and any $co-T^p$ -space is a G'_p -space.

COROLLARY 2.12. Let X be a G'_r -space for a map $r : X \rightarrow A$.

- (1) If $r : X \rightarrow A$ has a right homotopy inverse $i : A \rightarrow X$, then A is a G' -space.
- (2) If $r : X \rightarrow A$ has a left homotopy inverse $i : A \rightarrow X$, then X is a G' -space.

Proof. (1) It is sufficient to show that $H^n(A) = G^n(A)$ for all n . Since X is a G'_r -space, $H^n(X) = G^n(X, r, A)$. Moreover, we know, from the fact $ri \sim 1 : A \rightarrow A$, that $i^* : H^n(X) \rightarrow H^n(A)$ is an epimorphism. Thus we have, from Proposition 2.1(5), that $H^n(A) = i^*(H^n(X)) =$

$i^*(G^n(X, r, A)) \subset G^n(A, ri, A) = G^n(A)$ and A is a G' -space. (2) We show that $H^n(X) \subset G^n(X)$ for all n . We obtain, from Proposition 2.1(1), that $H^n(X) = G^n(X, r, A) \subset G^n(X, ir, X) = G^n(X, 1, X) = G^n(X)$. Thus we know that X is a G' -space. \square

From the above corollary, we know that if X dominates A and X is a G' -space, then A is also a G' -space. Moreover, we can clearly obtain, from Proposition 2.1(2),(3), the following corollary.

COROLLARY 2.13. *X is a G' -space if and only if for any space A and any map $p : X \rightarrow A$, X is a G'_p -space for a map $p : X \rightarrow A$.*

It is well known fact [20, Theorem 2.8] that $p : X \rightarrow A$ is a cocyclic map if and only if $DG(X, p, A; B) = [X, B]$ for any space B . It is also known [7, Proposition 15.8] that for any cofibration sequence $B \xrightarrow{i} E \xrightarrow{q} F \xrightarrow{\delta} \Sigma B \rightarrow \dots$, $\delta : F \rightarrow \Sigma B$ is cocyclic. Thus we have that for any cofibration sequence $B \xrightarrow{i} E \xrightarrow{q} F \xrightarrow{\delta} \Sigma B \rightarrow \dots$, $G^n(F, \delta, \Sigma B; \pi) = H^n(F; \pi)$ for all n . The cuplength, $cup(X)$, [12] is the length of the longest nontrivial product in the reduced cohomology $\tilde{H}^*(X)$. Let R be a ring. let $P^n(X; R) = \{\alpha \in H^n(X; R) | \beta \cup \alpha = 0 \text{ for all } \beta \in \tilde{H}^*(X, R)\}$. Then it is known [6] that $G^n(X; R) \subset P^n(X; R)$ for all n and R . Now we have an example which is a G'_p -space, but not a G' -space.

EXAMPLE 2.14. *Consider the complex projective space CP^2 . There is a cofibration of the unitary groups $U(2, 1) \xrightarrow{i} U(3) \xrightarrow{q} CP^2$ [17]. From the cofibration sequence $U(2, 1) \xrightarrow{i} U(3) \xrightarrow{q} CP^2 \xrightarrow{\delta} \Sigma U(2, 1) \rightarrow \dots$, we know that $G^n(CP^2, \delta, \Sigma U(2, 1)) = H^n(CP^2)$ for all n and CP^2 is a G'_δ -space. However, it is known [12] that $cup(CP^2) = 2$. Thus, from the the above fact of that $G^n(X) \subset P^n(X)$, we know that CP^2 is not a G' -space.*

3. G'_p -spaces for maps and homology decompositions

Given maps $p : X \rightarrow A$, $p' : X' \rightarrow A'$, let $(s, r) : p' \rightarrow p$ be a map from p' to p , that is, the following diagram is commutative;

$$\begin{array}{ccc} X' & \xrightarrow{p'} & A' \\ r \downarrow & & s \downarrow \\ X & \xrightarrow{p} & A. \end{array}$$

It is a well known fact that $Y \xrightarrow{\iota} cY \rightarrow \Sigma Y$ is a cofibration, where $\iota(y) = [y, 1]$. Let $i_r : X \rightarrow C_r$ be the cofibration induced by $r : X' \rightarrow X$ from $\iota_{X'} : X' \rightarrow cX'$. Let $i_s : A \rightarrow C_s$ be the cofibration induced by $s : A' \rightarrow A$ from $\iota_{A'} : A' \rightarrow cA'$. Then there is a map $\bar{p} : C_t \rightarrow C_s$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{p} & A \\ i_r \downarrow & & i_s \downarrow \\ C_r & \xrightarrow{\bar{p}} & C_s, \end{array}$$

where $C_r = cX' \amalg X/[x', 1] \sim r(x')$, and $C_s = cA' \amalg A/[a', 1] \sim s(a')$, $\bar{p} : C_r \rightarrow C_s$ is given by $\bar{p}([x', t]) = [p'(x'), t]$ if $[x', t] \in cX'$ and $\bar{p}(x) = p(x)$ if $x \in X$, $i_r(x) = x$, $i_s(a) = a$.

DEFINITION 3.1. Let X be a G'_p -space for a map $p : X \rightarrow A$. A map $(s, r) : p' \rightarrow p$ is called a G'_p -primitive if for each map $g : \Omega\Sigma X \rightarrow K(\mathbb{Z}, m)$, m arbitrary, there is a map $G : X \rightarrow A \vee K(\mathbb{Z}, m)$ such that $jG \sim (p \times g \circ e'_X)\Delta$ and $(i_s \vee 1)Gr \sim * : X' \rightarrow C_s \vee K(\mathbb{Z}, m)$, where $j : A \vee K(\mathbb{Z}, m) \rightarrow A \times K(\mathbb{Z}, m)$ is the inclusion and $e'_X : X \rightarrow \Omega\Sigma X$ is the adjoint functor image, $\tau(1_{\Sigma X})$, of $1_{\Sigma X}$.

The following lemmas are standard.

LEMMA 3.2. Let $f : X \rightarrow B$ be a map. Then there is a map $h : C_r \rightarrow B$ such that $hi_r = f$ if and only if $fr \sim *$.

LEMMA 3.3. [19] Let $g_t : C_r \rightarrow B_t (t = 1, 2)$ and $g : C_r \rightarrow B_1 \vee B_2$ be maps such that $p_t j g i_r \sim g_t i_r (t = 1, 2)$, where $j : B_1 \vee B_2 \rightarrow B_1 \times B_2$ is the inclusion and $p_t : B_1 \times B_2 \rightarrow B_t, t = 1, 2$ are projections. Then there is a map $h : C_r \rightarrow B_1 \vee B_2$ such that $g i_r = h i_r$ and $p_t j' h \sim g_t (t = 1, 2)$.

THEOREM 3.4. If X is a G'_p -space for a map $p : X \rightarrow A$ and $(s, r) : p' \rightarrow p$ is G'_p -primitive, then C_r is a $G'_{\bar{p}}$ -space for a map $\bar{p} : C_r \rightarrow C_s$.

Proof. It is sufficient to show that $H^m(C_r) \subset G^m(C_r, \bar{p}, C_s)$ for all m . For each m , let $f : C_r \rightarrow K(\mathbb{Z}, m) = \Omega K(\mathbb{Z}, m + 1)$ be any map. Then clearly we have the following homotopy commutative diagram;

$$\begin{array}{ccccc} X & \xrightarrow{i_r} & C_r & \xrightarrow{f} & \Omega K(\mathbb{Z}, m + 1) \\ e'_X \downarrow & & e'_{C_r} \downarrow & & \parallel \\ \Omega\Sigma X & \xrightarrow{\Omega\Sigma i_r} & \Omega\Sigma C_r & \xrightarrow{\Omega\tau^{-1}(f)} & \Omega K(\mathbb{Z}, m + 1). \end{array}$$

Since $(s, r) : p' \rightarrow p$ is a G'_p -primitive, for a map $\Omega\Sigma i_r \circ \Omega\tau^{-1}(f) : \Omega\Sigma X \rightarrow K(\mathbb{Z}, m)$, there is a map $G : X \rightarrow A \vee K(\mathbb{Z}, m)$ such that $jG \sim (p \times \Omega\Sigma i_r \circ \Omega\tau^{-1}(f) \circ e'_X)\Delta$ and $(i_s \vee 1)Gr \sim * : X' \rightarrow C_s \vee K(\mathbb{Z}, m)$, where $j : A \vee K(\mathbb{Z}, m) \rightarrow A \times K(\mathbb{Z}, m)$ is the inclusion and $e'_X : X \rightarrow \Omega\Sigma X$ is the adjoint functor image, $\tau(1_{\Sigma X})$, of $1_{\Sigma X}$. From Lemma 3.2, there is an extending $G' : C_r \rightarrow C_s \vee K(\mathbb{Z}, m)$ of $(i_s \vee 1) \circ G : X \rightarrow C_s \vee K(\mathbb{Z}, m)$, that is, $G' \circ i_r = (i_s \vee 1) \circ G$. Then we have that $p_1 j G' i_r = p_1 j (i_s \vee 1) G = p_1 (i_s \times 1) j G \sim p_1 (i_s \times 1) (p \times \Omega\Sigma i_r \circ \Omega\tau^{-1}(f) \circ e'_X)\Delta = i_s \circ p \sim \bar{p} \circ i_r$ and $p_2 j G' i_r = p_2 j (i_s \vee 1) G = p_2 (i_s \times 1) j G \sim p_2 (i_s \times 1) (p \times \Omega\Sigma i_r \circ \Omega\tau^{-1}(f) \circ e'_X)\Delta \sim \Omega\Sigma i_r \circ \Omega\tau^{-1}(f) \circ e'_X \sim f \circ i_r$. Thus we have, from Lemma 3.3, that there is a map $\bar{G} : C_r \rightarrow C_s \vee K(\mathbb{Z}, m)$ such that $\bar{G} i_r = G' i_r = (i_s \vee 1) G$ and $p_1 j' \bar{G} \sim \bar{p}$ and $p_2 j' \bar{G} \sim f$. Thus we know that $f : C_r \rightarrow K(\mathbb{Z}, m)$ is \bar{p} -cocyclic and C_r is a $G'_{\bar{p}}$ -space for a map $\bar{p} : C_r \rightarrow C_s$. \square

In 1959, Eckmann and Hilton [2] introduced a dual concept of Postnikov system as follows; A *homology decomposition of X* consists of a sequence of spaces and maps $\{X_n, q_n, i_n\}$ satisfying (1) $q_n : X_n \rightarrow X$ induces an isomorphism $(q_n)_* : H_i(X_n) \rightarrow H_i(X)$ for $i \leq n$, (2) $i_n : X_n \rightarrow X_{n+1}$ is a cofibration with cofiber $M(H_{n+1}(X), n)$ (a Moore space of type $(H_{n+1}(X), n)$), (3) $q_n \sim q_{n+1} \circ i_n$. It is known by [7] that if X be a 1-connected space having the homotopy type of CW complex, then there is a homology decomposition $\{X_n, q_n, i_n\}$ of X such that $i_n : X_n \rightarrow X_{n+1}$ is the principal cofibration induced from $\iota : M(H_{n+1}(X), n) \rightarrow cM(H_{n+1}(X), n)$ by a map $\kappa'_n : M(H_{n+1}(X), n) \rightarrow X_n$ which is called the dual Postnikov invariants. A space X is called a *rational space* [14] if X is a 1-connected space having homotopy type of a CW-complex such that for each $n > 0$, $H_n(X, \mathbb{Z})$ is a finite dimensional vector space over \mathbb{Q} . It is well known [14] that if X and A are rational spaces and $p : X \rightarrow A$ is a based map, then there exist homology decompositions $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ for X and A respectively and induced maps $\{p_n : X_n \rightarrow A_n\}$ satisfying

(1) for each n , the following diagram is homotopy commutative

$$\begin{array}{ccc}
 M(H_{n+1}(X), n) & \xrightarrow{\tilde{p}_*} & M(H_{n+1}(A), n) \\
 k'_n(X) \downarrow & & k'_n(A) \downarrow \\
 X_n & \xrightarrow{p_n} & A_n,
 \end{array}$$

that is, $(k'_n(A), k'_n(X)) : \tilde{p}_\# \rightarrow p_n$ is a map,

(2) $p_{n+1} : X_{n+1} \rightarrow A_{n+1}$ given by $p_{n+1} = \bar{p}_n$ satisfying commute diagram

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ i_n(=\iota_{k'_n(X)}) \downarrow & & i'_n(=\iota_{k'_n(A)}) \downarrow \\ X_{n+1} & \xrightarrow{p_{n+1}} & A_{n+1}, \end{array}$$

(3) for each n , the following diagram is homotopy commutative

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ q_n \downarrow & & q'_n \downarrow \\ X & \xrightarrow{p} & A. \end{array}$$

THEOREM 3.5. *Let X and A be rational spaces and $p : X \rightarrow A$ a map, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ homology decompositions for X and A respectively.*

- (1) *If X is a G'_p -space for a map $p : X \rightarrow A$, then each X_n is G'_{p_n} -space and the all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \rightarrow p_n$ are G'_{p_n} -primitive.*
- (2) *If X_n is a G'_{p_n} -space and the pair of k' -invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \rightarrow p_n$ is G'_{p_n} -primitive, then X_n is a G'_p -space.*

Proof. (1) Let $f : X_n \rightarrow K(\mathbb{Z}, m) = \Omega K(\mathbb{Z}, m+1)$ be any map. Since $(q_n)_* : H_i(X_n) \rightarrow H_i(X)$ for $i \leq n$ and $H_i(X_n) = 0$ for $i > n$, there is a map $f' : X \rightarrow K(\mathbb{Z}, m)$ such that $f'q_n \sim f$. Since X is a G'_p -space for a map $p : X \rightarrow A$, there is a map $G : X \rightarrow A \vee K(\mathbb{Z}, m)$ such that $jG \sim (p \times f')\Delta$, where $j : A \vee K(\mathbb{Z}, m) \rightarrow A \times K(\mathbb{Z}, m)$ is the inclusion. Let $\{B_n, q''_n, i''_n\}$ be a homology decomposition for $K(\mathbb{Z}, m)$. Then $\{A_n \vee B_n, q'_n \vee q''_n, i'_n \vee i''_n\}$ is a homology decomposition for $A \vee K(\mathbb{Z}, m)$. Then we have, by Toomer's result [15, Theorem 4], that there are families of maps $p_n : X_n \rightarrow A_n$ and $G_n : X_n \rightarrow A_n \vee B_n$ such that $i'_n p_n = p_{n+1} i_n$ and $q'_n p_n \sim p q_n$, and $(i'_n \vee i''_n) G_n = G_{n+1} i_n$ and $(q'_n \vee q''_n) G_n \sim G q_n$ for $n = 2, 3, \dots$ respectively, and $k'_n(A) \tilde{p}_* \sim p_n k'_n(X) : M(H_{n+1}(X), n) \rightarrow A_n$ and $(k'_n(A) \vee k'_n(K(\mathbb{Z}, m))) \tilde{G}_* \sim G_n k'_n(X) : M(H_{n+1}(X), n) \rightarrow A_n \vee B_n$, where $k'_n(A) : M(H_{n+1}(A), n) \rightarrow A_n$, $k'_n(X) : M(H_{n+1}(X), n) \rightarrow X_n$ and $k'_n(K(\mathbb{Z}, m)) : M(H_{n+1}(K(\mathbb{Z}, m)), n) \rightarrow B_n$ are k' -invariants of A , X and $K(\mathbb{Z}, m)$ respectively, $\tilde{p}_* : M(H_{n+1}(X), n) \rightarrow M(H_{n+1}(A), n)$ and $\tilde{G}_* : M(H_{n+1}(X), n) \rightarrow M(H_{n+1}(A \vee K(\mathbb{Z}, m)), n) \approx M(H_{n+1}(A \oplus H_{n+1}(K(\mathbb{Z}, m))), n) \approx M(H_{n+1}(A), n) \vee M(H_{n+1}(K(\mathbb{Z}, m)), n)$ are the induced maps by $p : X \rightarrow A$ and $G : X \rightarrow A \vee K(\mathbb{Z}, m)$ respectively. Consider the composition $G' = (1 \vee q''_n) G_n : X_n \rightarrow A_n \vee K(\mathbb{Z}, m)$. It is clear that $p_1 j G' = p_1 j G_n \sim p_n$ and $p_2 j G' = p_2 j (1 \vee q''_n) G_n \sim p_2 j G q_n =$

$f'q_n \sim f$. Thus $[f] \in G^m(X_n, p_n, A_n)$ and X_n is a G'_{p_n} -space for a map $p_n : X_n \rightarrow A_n$. Moreover, to show that $(k'_n(A), k'_n(X)) : \tilde{p}_* \rightarrow p_n$ is G'_{p_n} -primitive, let $g'' : \Omega \Sigma X_n \rightarrow K(\mathbb{Z}, m)$ be any map and m arbitrary. Since $g''e'_{X_n} : X_n \rightarrow K(\mathbb{Z}, m) \in H^m(X_n)$ is a map, by taking $f = g''e'_{X_n}$ from the above proof, we have a map $G' = (1 \vee q''_n)G_n : X_n \rightarrow A_n \vee K(\mathbb{Z}, m)$ such that $jG' \sim (p_n \times g''e'_{X_n})\Delta : X_n \rightarrow A_n \times K(\mathbb{Z}, m)$, where $j : A_n \vee K(\mathbb{Z}, m) \rightarrow A_n \times K(\mathbb{Z}, m)$ is the inclusion. Since $(i'_n \vee 1)G' = (i'_n \vee 1)(1 \vee q''_n)G_n = (i'_n \vee q''_n)G_n \sim (1 \vee q''_{n+1})G_{n+1}i_n : X_n \rightarrow A_{n+1} \vee K(\mathbb{Z}, m)$, $(i'_n \vee 1)G' : X_n \rightarrow A_{n+1} \vee K(\mathbb{Z}, m)$ has an extending $(1 \vee q''_{n+1})G_{n+1} : X_{n+1} \rightarrow A_{n+1} \vee K(\mathbb{Z}, m)$ and $(i'_n \vee 1)G'k'_n(X) \sim *$. Thus $(k'_n(A), k'_n(X)) : \tilde{p}_* \rightarrow p_n$ are G'_{p_n} -primitive.

(2) It follows from Theorem 3.4. □

Taking $p = 1_X$, $p' = 1_{X'}$, $s = r$, we can obtain the following corollary which is a dual result of Haslam's results about G -spaces and Postnikov systems [6].

COROLLARY 3.6. *Let X be rational space and $\{X_n, q_n, i_n\}$ homology decomposition for X .*

- (1) *If X is a G' -space, then each X_n is G' -space and all the k' invariants $k'_n(X)$ are G' -primitive.*
- (2) *If X_{n-1} is a G' -space and the k' -invariants $k'_n(X)$ is G' -primitive, then X_n is a G' -space.*

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